

SUPER YANGIAN DOUBLE $DY(gl(1|1))$ AND ITS GAUSS DECOMPOSITION

Jin-fang Cai, Guo-xing Ju, Ke Wu
Institute of Theoretical Physics, Academia Sinica,
Beijing, 100080, P. R. China

and

Shi-kung Wang
CCAST (World Laboratory), P.O. Box 3730, Beijing, 100080, P. R. China
and Institute of Applied Mathematics, Academia Sinica,
Beijing, 100080, P. R. China

Abstract

We extend Yangian double to super (or graded) case and give its Drinfel'd generators realization by Gauss decomposition.

Quantum algebras(in general meaning, it includes Yangians and quantum affine algebras etc.) are new algebraic structures discovered about ten years ago [1-4], they play important roles in the study of soluble statistical models and quantum fields theory. Quantum universal enveloping algebras of simple Lie algebras, which are related with some simple solutions (without spectral parameter) of Yang-Baxter equation, have been extensively and deeply studied in the past few years. Quantum affine algebras and Yangians are related respectively with trigonometric and rational solutions of Yang-Baxter equation, they have three realizations in literatures: Chevalley generators, T^\pm -matrix and Drinfel'd generators. The first realization was proposed independently by Drinfel'd and Jimbo, the second realization has direct meaning in quantum inverse scattering method and it's convenient to introduce central extension using this realization [5]. The isomorphism of T^\pm -matrix and Drinfel'd generators realizations of quantum affine algebra was established through Gauss decomposition by Ding and Frenkel [17]. The Yangian can be viewed as a deformation of only half of the corresponding loop algebra, while Yangian double are deformation of the complete loop algebra. Yangian doubles (and with central extension) have been studied by Bernard, Khoroshkin and Iohara etc.[6-12]. The

properties of super Yangians and its representations have been studied by some authors [13,14]. But to the author's knowledge, the super Yangian doubles and Gauss decomposition in super case have not been studied up to now. In this paper, we extend Yangian double to super case and give its Drinfel'd generators realization through Gauss decomposition.

Given a 2-dimensional graded vector space V and let its second basis be odd, then the super (graded) Yang-Baxter equation [15-17] takes this form:

$$\eta_{12}R_{12}(u)\eta_{13}R_{13}(u+v)\eta_{23}R_{23}(v) = \eta_{23}R_{23}(v)\eta_{13}R_{13}(u+v)\eta_{12}R_{12}(u), \quad (1)$$

where $R(u) \in \text{End}(V \otimes V)$ and it must obey weight conservation condition: $R_{ij,kl} \neq 0$ only when $i+j = k+l$. $\eta_{ik,jl} = (-1)^{(i-1)(k-1)}\delta_{ij}\delta_{lk}$. In dealing with tensor product in graded case, we must use this form: $(A \otimes B)(C \otimes D) = (-1)^{P(B)P(C)}AC \otimes BD$. The super Yang-Baxter equation can also be writed in components as follow:

$$\begin{aligned} & R_{ib,pr}(u)R_{pa,js}(u+v)R_{rs,dc}(v)(-1)^{(r-1)(s+a)} \\ &= (-1)^{(e-1)(f+c)}R_{ba,ef}(v)R_{if,kc}(u+v)R_{ke,jd}(u) \end{aligned} \quad (2)$$

It is easily to prove that $R(u) = uI + \hbar\mathcal{P}$ satisfy the above super Yang-Baxter equation, here $\mathcal{P}_{ik,jl} = (P\eta)_{ik,jl} = (-1)^{(i-1)(k-1)}\delta_{il}\delta_{jk}$ is the permutation operator in graded case.

In similiar to $DY(gl_2)$ [10], Super Yangian Double $DY(gl(1|1))$ is a Hopf algebra generated by $\{t_{ij}^k | 1 \leq i, j \leq 2, k \in \mathbf{Z}\}$ which obey the following relations:

$$\begin{aligned} R(u-v)T_1^\pm(u)\eta T_2^\pm(v)\eta &= \eta T_2^\pm(v)\eta T_1^\pm(u)R(u-v) \\ R(u-v)T_1^\pm(u)\eta T_2^\mp(v)\eta &= \eta T_2^\mp(v)\eta T_1^\pm(u)R(u-v) \end{aligned} \quad (3)$$

here we use standard notation: $T_1^\pm(u) = T^\pm(u) \otimes \mathbf{1}$, $T_2^\pm(u) = \mathbf{1} \otimes T^\pm(u)$ and $T^\pm(u) = (t_{ij}^\pm(u))_{1 \leq i, j \leq 2}$, $t_{ij}^\pm(u)$ are generate functions of t_{ij}^k :

$$t_{ij}^+(u) = \delta_{ij} - \hbar \sum_{k \geq 0} t_{ij}^k u^{-k-1}, \quad t_{ij}^-(u) = \delta_{ij} + \hbar \sum_{k < 0} t_{ij}^k u^{-k-1}. \quad (4)$$

The Hopf pairing relation between $T^+(u)$ and $T^-(v)$ is defined as:

$$\langle T_1^+(u), T_2^-(v) \rangle = R(u-v) \quad (5)$$

From relations (3), we can get the (anti-)commutation relations among $\{t_{ij}^k(u)\}$:

$$(u-v)[t_{ij}^\sigma(u), t_{kl}^\rho(v)] + (-1)^{ij+jk+ki+1}\hbar(t_{kj}^\sigma(u)t_{il}^\rho(v) - t_{kj}^\rho(v)t_{il}^\sigma(u)) = 0 \quad (6)$$

where (σ, ρ) take $(+, +)$, $(-, -)$ and $(+, -)$. The parity of t_{ij}^\pm equal to $(i-1)(j-1)$ which means that $(t_{11}^\pm(u), t_{22}^\pm(u))$ are even and $(t_{12}^\pm, t_{21}^\pm(u))$ are odd. We denote $[\ , \]$ as super commutator which is anti-commutator only when each of the elements in it is odd (or Fermionic) type. The follows are some special examples of the above relation:

$$\begin{aligned} (u-v)[t_{11}^\sigma(u), t_{12}^\rho(v)] + \hbar(t_{11}^\sigma(u)t_{12}^\rho(v) - t_{11}^\rho(v)t_{12}^\sigma(u)) &= 0 \\ (u-v)[t_{22}^\sigma(u), t_{12}^\rho(v)] - \hbar(t_{12}^\sigma(u)t_{22}^\rho(v) - t_{12}^\rho(v)t_{22}^\sigma(u)) &= 0 \\ (u-v)\{t_{12}^\sigma(u), t_{21}^\rho(v)\} - \hbar(t_{22}^\sigma(u)t_{11}^\rho(v) - t_{22}^\rho(v)t_{11}^\sigma(u)) &= 0 \end{aligned} \quad (7)$$

Hopf structure of $DY(gl(1|1))$ are defined as follows:

$$\begin{aligned} \Delta t_{ij}^\pm(u) &= \sum_{k=1,2} t_{kj}^\pm(u) \otimes t_{ik}^\pm(u) (-1)^{(k+i)(k+j)}, \\ \epsilon(t_{ij}^\pm(u)) &= \delta_{ij}, \quad S(stT^\pm(u)) = [stT^\pm(u)]^{-1}. \end{aligned} \quad (8)$$

here $[stT^\pm(u)]_{ij} = (-1)^{i+j}t_{ji}^\pm(u)$.

We can get the Drinfel'd generators realization of $DY(gl(1|1))$ by Gauss decomposition in the same way which has been used in $DY(gl_2)$. Introducing transformation for generate functions as follows:

$$\begin{aligned} E^\pm(u) &= \frac{1}{\hbar}t_{11}^\pm(u + \frac{\hbar}{2})^{-1}t_{12}^\pm(u + \frac{\hbar}{2}), \quad F^\pm(u) = \frac{1}{\hbar}t_{21}^\pm(u + \frac{\hbar}{2})t_{11}^\pm(u + \frac{\hbar}{2})^{-1}, \\ k_1^\pm(u) &= t_{11}^\pm(u), \quad k_2^\pm(u) = t_{22}^\pm(u) - t_{21}^\pm(u)t_{11}^\pm(u)^{-1}t_{12}^\pm(u). \end{aligned} \quad (9)$$

it means

$$\begin{aligned} T^\pm &= \begin{pmatrix} 1 & & \\ \hbar F^\pm(u - \frac{\hbar}{2}) & 1 & \\ k_1^\pm(u) & & \end{pmatrix} \begin{pmatrix} k_1^\pm(u) & & \\ & k_2^\pm(u) & \\ & & \hbar k_1^\pm(u)E^\pm(u - \frac{\hbar}{2}) \end{pmatrix} \begin{pmatrix} 1 & \hbar E^\pm(u - \frac{\hbar}{2}) & \\ & 1 & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} k_1^\pm(u) & & \\ \hbar F^\pm(u - \frac{\hbar}{2})k_1^\pm(u) & k_2^\pm(u) + \hbar^2 F^\pm(u - \frac{\hbar}{2})k_1^\pm(u)E^\pm(u - \frac{\hbar}{2}) & \\ & & \hbar k_1^\pm(u)E^\pm(u - \frac{\hbar}{2}) \end{pmatrix} \end{aligned} \quad (10)$$

and let

$$\begin{aligned} E(u) &= E^+(u) - E^-(u), \quad F(u) = F^+(u) - F^-(u), \\ H^\pm(u) &= k_2^\pm(u + \frac{\hbar}{2})k_1^\pm(u + \frac{\hbar}{2})^{-1}, \quad K^\pm(u) = k_2^\pm(u + \frac{\hbar}{2})k_1^\pm(u - \frac{\hbar}{2}). \end{aligned} \quad (11)$$

By the similiar calculate process which have been used in quantum affine algebra [18], we can obtain that these new generate functions $E(u), F(u), H^\pm(u)$ and $K^\pm(u)$ satisfy the following commutation relations:

$$\begin{aligned}
[H^\sigma(u), H^\rho(v)] &= [H^\sigma(u), K^\rho(v)], \quad \forall \sigma, \rho = +, - \\
[K^\sigma(u), K^\rho(v)] &= 0, \\
[H^\pm(u), E(v)] &= [H^\pm(u), F(v)] = 0, \\
E(v)K^\pm(u) &= \frac{u-v+\hbar}{u-v-\hbar} K^\pm(u)E(v), \\
F(v)K^\pm(u) &= \frac{u-v-\hbar}{u-v+\hbar} K^\pm(u)F(v), \\
\{E(u), E(v)\} &= \{F(u), F(v)\} = 0, \\
\{E(u), F(v)\} &= \frac{1}{\hbar} \delta(u-v)(H^-(v) - H^+(u)).
\end{aligned} \tag{12}$$

The coproduct and counit structure for $E(u), F(u), H^\pm(u)$ and $K^\pm(u)$ can be calculate out as follows:

$$\begin{aligned}
\Delta E^\pm(u) &= E^\pm(u) \otimes 1 + K^\pm(u) \otimes E^\pm(u) \\
\Delta F^\pm(u) &= 1 \otimes F^\pm(u) + F^\pm(u) \otimes k^\pm(u) \\
\Delta K^\pm(u) &= K^\pm(u) \otimes K^\pm(u) - F^\pm(u - \hbar) K^\pm(u) \otimes (E^\pm(u) K(u) + K(u) E^\pm(u)) \\
\Delta H^\pm(u) &= H^\pm(u) \otimes H^\pm(u) \\
\epsilon(K^\pm(u)) &= \epsilon(H^\pm(u)) = 1 \\
\epsilon(E^\pm(u)) &= \epsilon(F^\pm(u)) = 0
\end{aligned} \tag{13}$$

Expanding $H^\pm(u), K^\pm(u), E(u), F(u)$ in Fourier series

$$\begin{aligned}
E(u) &= \sum_{k \in \mathbb{Z}} e_k u^{-k-1}, \quad F(u) = \sum_{k \in \mathbb{Z}} f_k u^{-k-1}, \\
H^+(u) &= 1 + \hbar \sum_{k \geq 0} h_k u^{-k-1}, \quad H^-(u) = 1 - \hbar \sum_{k < 0} h_k u^{-k-1}, \\
K^+(u) &= 1 + \hbar \sum_{l \geq 0} k_l u^{-l-1}, \quad K^-(u) = 1 - \hbar \sum_{l < 0} k_l u^{-l-1}.
\end{aligned} \tag{14}$$

then the commutation relations among h_l, k_l, e_l and f_l give Drinfeld's generator realization of $DY(gl(1|1))$:

$$\begin{aligned}
[h_k, h_l] &= [k_k, k_l] = [h_k, k_l] = 0 \\
[h_k, e_l] &= [h_k, f_l] = 0 \\
[k_0, e_l] &= -2e_l, \quad [k_0, f_l] = 2f_l \\
[k_{k+1}, e_l] - [k_k, e_{l+1}] + \hbar(k_k e_l + e_l k_k) &= 0 \\
[k_{k+1}, f_l] - [k_k, f_{l+1}] - \hbar(k_k f_l + f_l k_k) &= 0 \\
\{e_k, f_l\} &= -2h_{k+l} \\
\{e_k, e_l\} &= \{f_k, f_l\} = 0
\end{aligned} \tag{15}$$

It is naturally that maybe we can also discuss central extension of $DY(gl(1|1))$ and the Gauss decomposition can also be applied to quantum affine superalgebra to get Ding-Frenkel map of its two realizations. As to its physical applicant, Yangian symmetry have been discovered in integrable quantum field [6,7], spin chains with long range interactions (including Calogero-Sutherland models) [19,20], Hubbard model [21], conformal field [22,23] and so on. Super Yangian has also been applied in color Calogero-Sutherland models [24] . Maybe the symmetries in these systems are the corresponding (super) Yangian doubles as ref.[6].

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